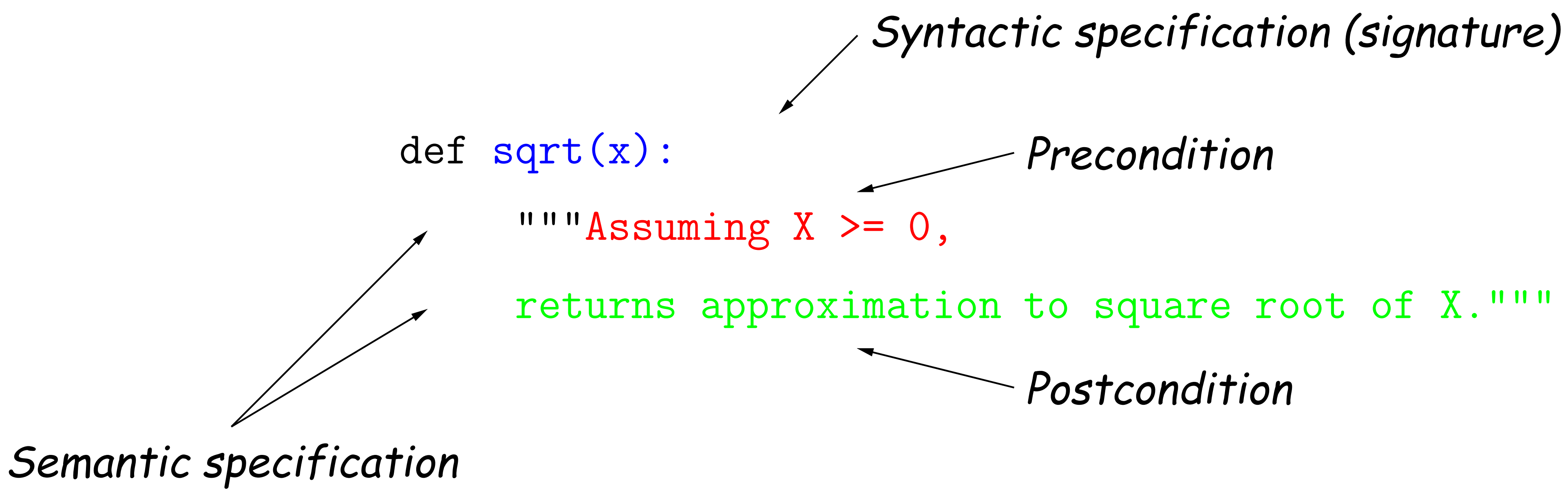
Announcements

- Computer-Science Mentors (CSM) will be opening section signups tonight (Monday, Jan. 30) at 8pm. Details will appear on Piazza.
- Starting this Friday, I'll start a series of extra lectures for those who want them, 4:30-6:00PM in 306 Soda, covering various topics we don't have room for. It is completely optional, and is *not* intended to help you with the course. Sign up for 1 unit of CS198 P/NP under CCN 34691 if interested. To get the unit, attendance required, and a few homeworks.
- HW 2 will be released today. Due next Monday.

Lecture #6: Recursion

Philosophy of Functions (I)



```
def sqrt(x):
    """Assuming X >= 0,
    returns approximation to square root of X."""
```

- Specifies a *contract* between caller and function implementor.
- **Syntactic specification** gives syntax for calling (number of arguments).
- **Semantic specification** tells what it does:
  - **Preconditions** are requirements on the caller.
  - **Postconditions** are promises from the function's implementor.

Philosophy of Functions (II)

- Ideally, the specification (syntactic and semantic) should suffice to use the function (i.e., without seeing the body).
- There is a *separation of concerns* here:
  - The caller (client) is concerned with providing values of x, a, b, and c and using the result, but *not* how the result is computed.
  - The implementor is concerned with how the result is computed, but not where x, a, b, and c come from or how the value is used.
  - From the client's point of view, `sqrt` is an *abstraction* from the set of possible ways to compute this result.
  - We call this particular kind *functional abstraction.*
- Programming is largely about choosing abstractions that lead to clear, fast, and maintainable programs.

Philosophy of Functions (III)

- Each function should have exactly one, logically coherent and well defined job.
  - Intellectual manageability.
  - Ease of testing.
- Functions should be properly documented, either by having names (and parameter names) that are unambiguously understandable, or by having comments (docstrings in Python) that accurately describe them.
  - Should be able to understand code that calls a function without reading the body of the function.
- Don't Repeat Yourself (*DRY*).
  - Simplifies revisions.
  - Isolates problems.

Philosophy of Functions (IV)

- Corollary of DRY: Make functions general
  - copy-paste leads to maintenance headaches
- Taking two (nearly) repeated sections of program code and turning them into calls to a common function is an example of *refactoring*.
- Keep names of functions and parameters meaningful:

Instead of	**Use**
boolean	turn_is_over
d	dice
helper	take_turn

(Bowling example From Kernighan&Plauger):

y	score
L	ball
f	frame

Simple Linear Recursions (Review)

- We've been dealing with recursive function (those that call themselves, directly or indirectly) for a while now.
- From Lecture #3:

```
def sum_squares(N):
    """The sum of K**2 for K from 1 to N (inclusive)."""
    if N < 1:
        return 0
    else:
        return N**2 + sum_squares(N - 1)
```

- This is a simple *linear recursion*, with one recursive call per function instantiation.
- Can imagine a call as an expansion:

```
sum_squares(3) => 3**2 + sum_squares(2)
               => 3**2 + 2**2 + sum_squares(1)
               => 3**2 + 2**2 + 1**2 + sum_squares(0)
               => 3**2 + 2**2 + 1**2 + 0 => 14
```

- Each call in this expansion corresponds to an environment frame, linked to the global frame, as shown here.

Tail Recursion

- We've also seen a special kind of linear recursion that is strongly linked to iteration:

```
def sum_squares(N):
    """The sum of K**2
    for 1 <= K <= N."""
    accum = 0
    k = 1
    while k <= N:
        accum += k**2
        k += 1
    return accum
```

```
def sum_squares(N):
    """The sum of K**2
    for 1 <= K <= N."""
    def part_sum(k, accum):
        if k <= N:
            return part_sum(k+1, accum + k**2)
        else:
           return accum
    part_sum(1, 0)
```

- The right version is a *tail-recursive function*: the recursive call is either the returned value or very last action performed.
- The environment frames correspond to the iterations of the loop on the left, as shown here.

Recursive Thinking

- So far in this lecture, I've shown recursive functions by tracing or repeated expansion of their bodies.
- But when you call a function from the Python library, you don't look at its implementation, just its documentation ("the contract").
- *Recursive thinking* is the extension of this same discipline to functions *as you are defining them.*
- When implementing `sum_squares`, we reason as follows:
  - **Base case:** We know the answer is 0 if there is nothing to sum ($N < 1$).
  - Otherwise, we observe that the answer is N^2 plus the sum of the positive integers from 1 to $N - 1$.
  - But there is a function (`sum_squares`) that can compute $1 + \ldots + N - 1$ (its comment says so).
  - So when $N \geq 1$, we should return $N^2 + \texttt{sum\_squares}(N - 1)$.
- This "recursive leap of faith" works as long as we can guarantee we'll hit the base case.

Recursive Thinking in Mathematics

- To prevent an infinite recursion, must use this technique only when
  - The recursive cases are “smaller” than the input case, and
  - There is a minimum “size” to the data, and
  - All chains of progressively smaller cases reach a minimum in a finite number of steps.
- We say that such “smaller than” relations are *well founded*.
- We have

  Theorem (Noetherian Induction): Suppose $\prec$ is a well-founded relation and P is some property (predicate) such that whenever $P(y)$ is true for all $y \prec x$, then $P(x)$ is also true. Then $P(x)$ is true for all x.

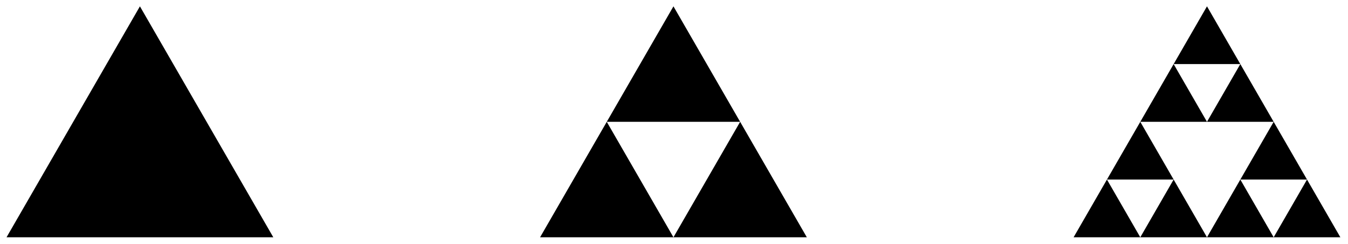
  (After Emmy Noether 1882–1935, Göttingen and Bryn Mawr).
- More general than the “line of dominos” induction you may have encountered: If true for a base case b, and if true for k when true for $k-1$, then true for all $k > b$.

A Problem

```
def find_first(start, pred):
    """Find the smallest k >= START such that PRED(START)."""
    ?
```

Subproblems and Self-Similarity

- Recursive routines arise when solving a problem naturally involves solving smaller instances of the same problem.
- A classic example where the subproblems are visible is *Sierpinski's Triangle* (aka bit Sierpinski's Gasket).
- This triangle may be formed by repeatedly replacing a figure, initially a solid triangle, with three quarter-sized images of itself (1/2 size in each dimension), arranged in a triangle:



- Or we can think creating a "triangle of order N and size S" by drawing either
  - a solid triangle with side S if $N = 0$, or
  - three triangles of size $S/2$ and order $N - 1$ arranged in a triangle.

The Gasket in Python

- We can describe this as a recursive Python program that produces Postscript output.

```
sin60 = sqrt(3) / 2
def make_gasket(x, y, s, n, output):
    """Write Postscript code for a Sierpinski's gasket of order N
    with lower-left corner at (X, Y) and side S on OUTPUT."""
    if n == 0:
        draw_solid_triangle(x, y, s, output)
    else:
        make_gasket(x, y, s/2, n - 1, output)
        make_gasket(x + s/2, y, s/2, n - 1, output)
        make_gasket(x + s/4, y + sin60*s/2, s/2, n - 1, output)

def draw_solid_triangle(x, y, s, output):
    "Draw a solid triangle lower-left corner at (X, Y) and side S."
    print("{x} {y} moveto "   # Go x, y
          "{s} 0 rlineto "    # Horizontal move by s units
          "-{mid} {alt} rlineto " # Move up and to left
          "closepath fill"    # Close path and fill with black
          .format(x=x, y=y, s=s, mid=s/2, alt=s*sin60), file=output)
```

Aside: Using the Functions

- Just to complete the picture, we can use `make_gasket` to create a standalone Postscript file on a given file.

```
def draw_gasket(n, output=sys.stdout):
    print("%!", file=output)
    make_gasket(100, 100, 400, 8, output)
    print("showpage", file=output)
    output.flush()  # Make sure all output so far is written
```

- And just for fun, here's some Python magic to display triangles automatically (uses `gs`, the Ghostscript interpreter for Postscript).

```
from subprocess import Popen, PIPE, DEVNULL

def make_displayer():
    """Create a Ghostscript process that displays its input (sent in through
    .stdin)."""
    return Popen("gs", stdin=PIPE, stdout=DEVNULL)

>>> d = make_displayer()
>>> draw_gasket(5, d.stdin)
>>> draw_gasket(10, d.stdin)
```

Aside: The Gasket in Pure Postscript

- One can also perform the logic to generate figures in Postscript directly, which is itself a full-fledged programming language:

```
%!

/sin60 3 sqrt 2 div def

/make_gasket {
    dup 0 eq {
3 index 3 index moveto 1 index 0 rlineto 0 2 index rlineto
        1 index neg 0 rlineto closepath fill
    } {
3 index 3 index 3 index 0.5 mul 3 index 1 sub make_gasket
3 index 2 index 0.5 mul add 3 index 3 index 0.5 mul
      3 index 1 sub make_gasket
3 index 2 index 0.25 mul add 3 index 3 index 0.5 mul add
      3 index 0.5 mul 3 index 1 sub make_gasket
    } ifelse
    pop pop pop pop
} def

100 100 400 8 make_gasket showpage
```